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The Notion of Neatness in Torsion Abelian Groups of Finite Rank

by

William Yslas VÉLEZ*

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Throughout this paper we shall use the notation and terminology of [1]. Let G be a torsion abelian group of finite rank with $G=\bigoplus_p G_p$, where G_p denotes the subgroup of elements whose orders are powers of a prime p and

$$r(G)=\sum_p r_p(G)=\sum_p r(G_p)\,.$$

A subgroup H of G is said to be a neat subgroup if $pH=pG\cap H$, for all primes p. The notion of neatness was introduced by K. Honda (see Page 131 of [1]). The following theorem provides a characterization of this concept for the groups in question.

THEOREM. *Let G be a torsion abelian group of finite rank and H a subgroup of G. Then H is a neat subgroup of G iff for every subgroup S of H, $H\neq S$, $r(G/S)>r(G/H)$.*

Proof. Let $H=\bigoplus_p H_p$, $S=\bigoplus_p S_p$ be the decomposition into p-groups, then H is neat in G iff H_p is neat in G_p, for every prime p. Let us suppose that the theorem is true for p-groups. Let H be neat in G and S a subgroup of H with $S\neq H$. Then H_p is neat in G_p, for all p, thus if $S_p\neq H_p$, we have that $R(G_p/S_p)>r(G_p/H_p)$. Since $S\neq H$, there exists at least one p with $S_p\neq H_p$, so

$$r(G/S)=\sum_p r(G_p/S_p)>\sum_p r(G_p/H_p)=r(G/H)\,.$$

Conversely, assume that $R(G/S)>r(G/H)$ for every subgroup S of H, $S\neq H$. Fix a prime p and let S' be any subgroup of H_p with $S'\neq H_p$. Let $S=\bigoplus_q S_q$, where q runs through all the primes and $S_q=H_q$, for $q\neq p$ and $S_p=S'$. Then S is a subgroup of H, with $S\neq H$, thus $r(G/S)>r(G/H)$, which implies that $r(G_p/S')>r(G_p/H_p)$, and since we are assuming the theorem is true for p-groups, H_p is neat in G_p, for each prime p, thus H is neat in G.

Thus to prove the theorem we may assume that G is a p-group. Before proving

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the theorem for p-groups we make two observations.

Let D be the maximal divisible subgroup of G, then for any subgroup T of G, it is an easy calculation to verify that the maximal divisible subgroup of G/T is $(D+T)/T$.

If $S<H<G$, with H neat in G, then again it is easy to see that H/S is neat in G/S.

Let

$$G/S=(D+S)/S \oplus\left(\bigoplus_{i=1}^{k} \langle b_i+S\rangle\right),$$

where D is the maximal divisible subgroup of G and $\langle b_i+S\rangle$ is each a finite cyclic group. Thus $r(G/S)=r(D+S)/S)+k$. Let $h\in H$, then

$$h+S=\left(d+\sum_{i=1}^{k} n_ib_i\right)+S,$$

where $d\in D$, $n_i\in Z$. If $p\nmid n_i$ for some i (say $i=k$), then

$$b_k+H\in(D+H)/H\oplus\left(\bigoplus_{i=1}^{k-1} \langle b_i+H\rangle\right)$$

so $r(G/H)\leq r((D+H)/H)+k-1<r((D+S)/S)+k=r(G/S)$ and we would be done. Thus, we may assume that for all $h\in H$, and

$$h+S=\left(d+\sum_{i=1}^{k} n_ib_i\right)+S,$$

we have that $p\mid n_i$, for all i. Now, $d\in D$ implies that $d=pd_1$, $d_1\in D$, thus

$$h+S=p\left(\left(d_1+\sum_{i=1}^{k} (n_i/p)b_i\right)+S\right)\in p(G/S),$$

thus $H/S\subset p(G/S)$. However, since H/S is neat in G/S, we have that $p(H/S)=p(G/S)\cap(H/S)$. But we have just shown that $H/S\subset p(G/S)$, thus we have that $p(H/S)=H/S$ is divisible and is thus a summand of G/S. So we have that $G/S\cong(H/S)\oplus A$, where $A\cong(G/S)/(H/S)\cong G/H$ and thus $r(G/S)+r(G/H)>r(G/H)$ since $S\neq H$.

Now to prove the converse. Assume that whenever $S<H<G$, $S\neq H$, we have that $r(G/S)>r(G/H)$.

Let

$$H=D_1\oplus\left(\bigoplus_{i=1}^{h} \langle a_i\rangle\right).$$

where D_1 is divisible and $\langle a_i\rangle$ is a finite cyclic group. Suppose that $pb\in H$, $b\in G$. Thus

$$pb=d+\sum_{i=1}^{h} n_ia_i,\ d\in D.$$

If $p\mid n_i$, for all i, then since $d=pd_1$,

$$pb=p\left(d_1+\sum_{i=1}^{h}(n_i/p)a_i\right)=pa,\ a\in H\ .$$

Hence, by way of contradiction, assume that $p\nmid n_i$ for at least one i. By reordering if necessary, assume that $p\nmid n_i$, $i=1, \cdots, t$, $p\mid n_i$, $i=t+1, \cdots, d$, $0(a_1)\geq 0(a_2)\geq\cdots\geq 0(a_t)$ (where $0(a)$ denotes the order of the element a) and $n_1=1$ (replace a_1 by $a_1{}^{n_1}$).

Set

$$b_1=b-d_1-\sum_{i=t+1}^{h}(n_i/p)a_i\,,$$

where $pd_1=d$. Then $pb_1=a_1'=a_1+n_2a_2+\cdots+n_ta_t$. It is easy to see that $\langle a_1, \cdots, a_t\rangle=\langle a_1', a_2, \cdots, a_t\rangle$ and since $0(a_1')\geq 0(a_i)$, $i=1, \cdots, t$, we also have that $\{a_1', a_2, \cdots, a_t\}$ are independent. Thus

$$\langle a_1, \cdots, a_t\rangle=\langle a_1'\rangle\oplus\left(\bigoplus_{i=2}^{t}\langle a_i\rangle\right).$$

Hence, we may assume that there is a $b\in G$ with $pb=a_1$, where

$$H=D_1\oplus\left(\bigoplus_{i=1}^{h}\langle a_i\rangle\right).$$

Set $S=D_1\oplus\langle pa_1\rangle\oplus\langle a_2\rangle\oplus\cdots\oplus\langle a_h\rangle$. Clearly $S\neq H$. Let

$$G/H=(D+H)/H\oplus\left(\bigoplus_{i=1}^{k}\langle b_i+H\rangle\right),$$

where $\langle b_i+H\rangle$ are all finite and D is the maximal divisible subgroup of G. Thus $r(G/H)=r((D+H)/H)+k=r(D)-r(D_1)+k$. Now it is obvious that $r((D+S)/S)=r(D)-r(D_1)$ and G/H is generated by $\{d+S:d\in D\}$, a_1+S, b_1+S, $\cdots$, b_k+S. However, there is a $b\in G$ with $pb=a_1$, thus

$$b+H=d+\sum_{i=1}^{k}n_ib_i+H,\ d\in D\,,$$

so

$$b=d+\sum_{i=1}^{k}n_ib_i+\sum_{i=1}^{h}m_ia_i+d',\ d'\in D_1$$

since

$$H=D_1\oplus\left(\bigoplus_{i=1}^{h}\langle a_i\rangle\right)$$

and $D_1\subset D$, thus

$$a_1 = pb = pd + \sum_{i=1}^{k} n_i pb_i + \sum_{i=1}^{h} m_i pa_i + pd' .$$

However, pd', $pa_i \in S$, so

$$a_1 + S = \left(pd + \sum_{i=1}^{k} n_i pb_i \right) + S ,$$

thus G/S is generated by $\{d+S : d \in D\}$, $b_1 + S, \cdots, b_k + S$, so $r(G/S) \leq r(D) - r(D_1) + k = r(G/H)$ which contradicts the assumption that $r(G/S) > r(G/H)$. ■

For a related result see Problem 14, Pg. 132, of [1].

Reference

[1] FUCHS, L.; *Infinite Abelian Groups*, Vol..I, Academic Press, New York and London, 1970.

Department of Mathematics
University of Arizona
Tucson, Arizona 85721
U.S.A.